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SHORTER COMMUNICATIONS

ASYMPTOTIC SOLUTIONS OF THE LAMINAR BOUNDARY-LAYER EQUATIONS*

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NOMENCLATURE

C, constant pressure gradient term in boundary-layer equations;
g, x-component function occurring in similarity variable;
h, y-component function occurring in the transformed coordinate;
M_{mn}, arbitrary function of ξ obtained by the nth integration of the mth order perturbation of the boundary-layer equation;
u, x-component of boundary-layer velocity;
U, freestream velocity;
v, y-component of boundary-layer velocity;
x, coordinate parallel to body surface measured from stagnation point;
y, coordinate normal to body surface.

Greek symbols
ε, the perturbation variable, $\varepsilon = \mathrm{d}\xi/\mathrm{d}x$;
η, transformed coordinate;
ν, kinematic viscosity;
ξ, transformed coordinate;
ψ, stream function.

INTRODUCTION

THE 'WEDGE' method for obtaining locally similar solutions of the boundary-layer equation has been known and used successfully for some time. This method involves the transformation of an independent variable, which produces a perturbation about a similar solution—the Falkner–Skan solution [1]. The transformation was first presented in this sense by Meksyn [2] and Gortler [3]. Merk [4] was able to form the perturbation series in terms of a universal function but obtained a solution for the first term of the series only. Chao and Fagbenle [5] were able to carry the expansion to four terms and produced very accurate solutions for flow and heat transfer over 2-dim. and axisymmetric bodies. It has not been known, however, how to obtain such transformations in general, the only existing one apparently having been obtained intuitively. In what follows, a general method for finding such transformations is presented, and a perturbation is made about a new similar solution to the laminar, incompressible, boundary-layer momentum equation. This similar solution is in terms of simple analytical functions, as are the higher order terms. The solution allows arbitrary accelerated edge velocity distribution and arbitrary wall transpiration.

The general applicability of the method is demonstrated through series solutions to the general form of the Riccati equation. Where comparisons are available, solutions match exactly those obtained by conventional methods. Further details and examples can be obtained in the original thesis [6].

*This work is based upon the author's doctoral thesis, University of Illinois.

SOLUTIONS OF THE BOUNDARY-LAYER EQUATION

Application of the transformation $\psi = \psi(\eta)$, $\eta = g(x) + h(y)$ to the steady, laminar, 2-dim. incompressible boundary-layer equation yields

$$\psi''' + \frac{u_e}{\nu} u_e' = 0 \tag{1}$$

where the primes on the stream function, ψ, indicate differentiation with respect to η, those on u_e, differentiation with respect to x. To obtain equation (1), it is required that $\eta = g(x) + y$, where the boundary conditions then determine that $g(x) = \delta$, the boundary-layer thickness. The boundary-layer then lies between the wall at $\eta = 0$ and the boundary-layer edge at $\eta = g(x)$, $y = 0$. Equation (1), which is linear in ψ, yields an ordinary differential equation only for the special case of a constant pressure gradient, for which case its solution yields accurate results. It is now desired to perturb equation (1) in order to increase its range of application. This is done by defining a new transformation of the form $\psi = \psi(\eta, \xi)$, where $\xi = \xi(x)$ and the functional form of ξ is as yet undetermined. The boundary-layer equation now becomes

$$\frac{\mathrm{d}\xi}{\mathrm{d}x}\left(\psi'\frac{\partial \psi'}{\partial \xi} - \psi''\frac{\partial \psi}{\partial \xi}\right) = u_e u_e' + \nu\psi'''. \tag{2}$$

The LHS of equation (2) is multiplied by $\mathrm{d}\xi/\mathrm{d}X$, which can now be used as a 'perturbation variable', $\varepsilon(\xi)$, thus eliminating the non-linear terms from the zero-order equation when a perturbation solution is assumed. This is the essential idea of the method—to choose an initially arbitrary transformation that then becomes the perturbation 'parameter'. Thus, in the usual manner we now set

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots \tag{3}$$

and also

$$u_e u_e' = C + \varepsilon, \tag{4}$$

i.e. the pressure gradient differs by an amount ε from being constant. ξ is now defined by equation (4) and by $\varepsilon = \mathrm{d}\xi/\mathrm{d}x$. Substitution of these equations into equation (2) and collecting terms of order-zero in ε give

$$\frac{\partial^3 \psi_0}{\partial \eta^3} + \frac{C}{\nu} = 0. \tag{5}$$

Equation (5) is identical to the previous similar solution, equation (1), except that ψ is now a function of two independent variables, η and ξ. However, since differentiation is only with respect to η, equation (5) may be integrated directly to give

$$\psi_0 = -\frac{C\eta^3}{6\nu} + \frac{M_{01}}{2}\eta^2 + M_{02}\eta + M_{03} \tag{6}$$

where the M_{mn} are arbitrary functions of ξ. The requirement that $\psi_0(\eta = 0) = 0$ gives $M_{03} = 0$. The velocity at the wall is determined by M_{02}, which would normally be zero but can also be used to allow arbitrary slip flow as a function of ξ (or x). The requirement that

$$\left.\frac{\partial u}{\partial y}\right|_e = 0 \text{ gives}$$

$$\left.\frac{\partial u}{\partial y}\right|_e = \left.\frac{\partial^2\psi_0}{\partial\eta^2}\right|_e = -\frac{C}{v}g + M_{01}. \tag{7}$$

The edge velocity is given by

$$u_e = \left.\frac{\partial\psi_0}{\partial\eta}\right|_e = -\frac{C}{2v}g^2 + M_{01}g. \tag{8}$$

Using equation (7) and solving equation (8) for g gives

$$g = \left(\frac{2vu_e}{C}\right)^{1/2}. \tag{9}$$

This is the boundary-layer thickness, but an appropriate value for C (the 'similar' pressure gradient that is being perturbed) has still to be determined.

The velocity gradient at the wall is given by

$$\left.\frac{\partial u}{\partial y}\right|_w = \left.\frac{\partial^2\psi_0}{\partial\eta^2}\right|_w = \frac{Cg}{v}. \tag{10}$$

This extremely simple zero-order solution already produces reasonable accuracy, but comparison will await the full solution.

The zero-order velocity in the y direction is obtained by differentiating equation (6) with respect to x

$$v_0 = \frac{dg}{dx}\left(\frac{C}{2v}\eta^2 - M_{01}\eta - M_{02}\right) - \varepsilon\left(\frac{\eta^2}{2}\frac{dM_{01}}{d\xi} + \frac{\eta dM_{02}}{d\xi} + \frac{dM_{03}}{d\xi}\right). \tag{11}$$

Evaluated at the wall, equation (11) gives

$$v_{0_w} = -\varepsilon\frac{dM_{03}}{d\xi}. \tag{12}$$

Thus, arbitrary wall blowing or suction as a function of ξ may be introduced by setting

$$M_{03} = -\int\frac{v_0}{\varepsilon}d\xi. \tag{13}$$

The order-one equation for ψ is obtained from equation (2) and (3)

$$v\psi_1''' + 1 = \psi_0'\frac{\partial\psi_0'}{\partial\xi} - \psi_0''\frac{\partial\psi_0}{\partial\xi} + \frac{d\varepsilon}{d\xi}(\psi_0'\psi_1' - \psi_0''\psi_1), \tag{14}$$

The terms multiplied by $d\varepsilon/d\xi$ immediately present a problem in that their retention greatly complicates the solution of equation (14), and even though it is linear in ψ_1, this writer was not able to find a general solution using conventional methods. A solution can be obtained by applying the method of this work specifically to equation (14) (i.e. by performing a separate perturbation), but the increased accuracy is not felt to warrant the increased complexity. It is thus now assumed that $d\varepsilon/d\xi \sim O(\varepsilon)$, and these terms are neglected in equation (14). Substitution of the previous solution for ψ_0 into the remaining terms of equation (14) gives

$$v\psi_1''' = \frac{M_{01}}{2}\frac{dM_{01}\eta^2}{d\xi} - \frac{dM_{03}}{d\xi}\left(M_{01} - \frac{C\eta}{v}\right) - 1. \tag{15}$$

Once again differentiation of the dependent variable is only with respect to η. Integration of equation (15) gives

$$v\psi_1 = \frac{M_{01}}{120}\frac{dM_{01}\eta^5}{d\xi} - \frac{dM_{03}}{d\xi}\left(\frac{M_{01}\eta^3}{6} - \frac{C\eta^4}{24v}\right) - \frac{\eta^3}{6} + \frac{M_{11}\eta^2}{2} + M_{12}\eta + M_{13}. \tag{16}$$

For this and subsequent orders, all of the boundary conditions are zero, which requires that $M_{12} = M_{13} = 0$. $M_{11}(\xi)$ is used to ensure that the order-one contribution to the edge velocity is zero. Thus, setting ψ_1' equal to zero at the boundary-layer edge ($\eta = g$) gives

$$M_{11} = \frac{g}{2} - \frac{M_{01}}{24}\frac{dM_{01}}{d\xi}g^3 + \frac{dM_{03}}{d\xi}\left(\frac{Cg^2}{6v} - \frac{M_{01}g}{2}\right). \tag{17}$$

Differentiating the solution, equation (16), and evaluating the result at $\eta = 0$ shows that

$$M_{11} = v\psi_1''\Big|_{\eta=0} = v\left.\frac{\partial u_1}{\partial y}\right|_w \tag{18}$$

i.e. M_{11} is the first-order contribution to the velocity gradient at the wall.

The boundary condition

$$\left.\frac{\partial^2\psi_1}{\partial y^2}\right|_e = 0 \tag{19}$$

requires that

$$C = u_e u_e'/2. \tag{20}$$

Of course, C is arbitrary but constant and thus can only satisfy the boundary condition (19) locally. Note that C could have been initially chosen as $C(\xi)$, and this choice indeed produces rapid convergence of the perturbation series, but the solution then becomes singular for some regions of interest.

The second-order solution is obtained in the same manner as the previous solutions. Only the results are presented here

$$vu_2 = v\frac{\partial\psi_2}{\partial\eta} = M_{21}\eta - \frac{M_{11}}{2v}\frac{dM_{03}\eta^2}{d\xi} + \left[\frac{dM_{03}}{d\xi} + M_{01}\left(\frac{dM_{03}}{d\xi}\right)^2\right]\frac{\eta^3}{6v} + \left[M_{11}\frac{dM_{01}}{d\xi} + M_{01}\frac{dM_{11}}{d\xi} - \frac{C}{v}\left(\frac{dM_{03}}{d\xi}\right)^2\right]\frac{\eta^4}{24v} - \left(\frac{M_{01}}{2}\frac{dM_{01}}{d\xi}\frac{dM_{03}}{d\xi} + \frac{M_{01}^2}{3}\frac{d^2M_{03}}{d\xi^2}\right)\frac{\eta^5}{20v} + \frac{CM_{01}}{144}\frac{d^2M_{03}}{d\xi^2}\frac{\eta^6}{v^2} + \left[\frac{M_{01}^2}{1260}\frac{d^2M_{01}}{d\xi^2} - \frac{M_{01}}{5040}\left(\frac{dM_{01}}{d\xi}\right)^2 - \frac{C^2}{1008v^2}\frac{d^2M_{03}}{d\xi^2}\right]\frac{\eta^7}{v} \times\left[\left(\frac{dM_{01}}{d\xi}\right)^2 + M_{01}\frac{d^2M_{01}}{d\xi^2}\right]\frac{C\eta^8}{4480v^2}. \tag{21}$$

Here, the boundary conditions have been evaluated exactly as in the first-order solution, except that C has already been used in the first-order solution to satisfy the boundary condition (19) and is no longer available to the second-order solution. This condition is thus left unsatisfied and limits the accuracy of the method to the three terms already obtained. The second-order velocity gradient at the wall is given by

$$\left.\frac{\partial u_2}{\partial y}\right|_{\eta=0} = \frac{M_{21}}{v} \tag{22}$$

where M_{21} is obtained in the same manner as M_{11}.

These expressions appear quite cumbersome but simplify greatly when substitutions for M_{01}, M_{11} etc. are made. Thus, when the M_{nm}'s and g are written in terms of u_e as defined

Table 1. Comparison of skin friction results for flow over a wedge with those of Hartree [7]

		$(C_f/2)\,Re_x^{1/2}$				
β	C^*	2 Terms	Hartree	% Difference	3 Terms	% Difference
0.1	Case 1	0.306	0.426	−28	0.434	1.9
	Case 2	0.297	0.426	−30		
0.3	Case 1	0.560	0.594	−5.7	0.585	−1.5
	Case 2	0.545	0.594	−8.4		
2/3	Case 1	0.943	0.899	4.9	0.888	−1.1
	Case 2	0.917	0.899	2.0		
1.0	Case 1	1.333	1.233	8.1	1.219	−1.1
	Case 2	1.296	1.233	5.2		
1.6	Case 1	2.667	2.405	10.9	2.382	−0.9
	Case 2	2.593	2.405	7.8		

*Case 1: $C = (ue/2)u_e'$; Case 2: $\varepsilon = 0 \Rightarrow C = u_e u_e'$.

previously, the zero-order solutions for velocity and velocity gradient at the wall, obtained by differentiating equation (6) become

$$u_0 = \frac{-C}{2\nu}\eta^2 + \left(\frac{2Cu_e}{\nu}\right)^{1/2}\eta \tag{23}$$

and

$$\left.\frac{\partial u_0}{\partial y}\right|_w = \left(\frac{2Cu_e}{\nu}\right)^{1/2}. \tag{24}$$

The corresponding first-order solutions are obtained from equation (16)

$$\varepsilon u_1 = \frac{1}{\nu}\left\{\frac{C}{24\nu}u_e'\eta^4 - \frac{\varepsilon\eta^2}{2} + \left[\varepsilon\left(\frac{\nu u_e}{2C}\right)^{1/2} - \frac{u_e^2}{6}\left(\frac{\nu}{2Cu_e}\right)^{1/2}u_e'\right]\eta - v_w\left[\left(\frac{Cu_e}{2\nu}\right)^{1/2}\eta^2 - \frac{C\eta^3}{6\nu}\right]\right\} \tag{25}$$

and

$$\varepsilon\left.\frac{\partial u_1}{\partial y}\right|_w = \left(\frac{u_e}{2\nu C}\right)^{1/2}\left(\varepsilon - \frac{u_e}{6}u_e'\right) - \frac{2}{3}\,\frac{v_w u_e}{\nu}. \tag{26}$$

Similar reductions are obtained in the second-order solutions.

All of the terms reduce even further when a suitable choice for the value of C is made. There are three useful choices. The first, as mentioned earlier, is $C = u_e u_e'/2$, which allows local satisfaction of the boundary condition $\partial u_1/\partial y_e = 0$. This produces the most accurate results in a three-term expansion but converges slowly. Another selection is to allow ε to go to zero locally, which implies $C = u_e u_e'$. Since this produces the smallest possible perturbation parameter, convergence is (usually) most rapid, with two terms producing useful results. The third term, however, diverges. Still another choice is necessary. Both of the above require C to contain u_e', which produces a singularity in the second-order terms when u_e' approaches zero, i.e. a low-pressure gradient flow. Since C is completely arbitrary (the perturbation can be about any similar solution), it is possible to select a value for C that shifts the singularity away from the point of zero pressure gradient. The approach taken here is to simply shift the singularity to the stagnation point by substituting $C = u_e^2/2$ for $C = u_e u_e'/2$. Note that the dimensions for this new magnitude of C must still be taken as length/(time)^2. This selection yields reasonable accuracy in the low-pressure gradient region for two terms but diverges with three terms because of loss of boundary condition satisfaction.

COMPARISONS WITH PREVIOUS SOLUTIONS

The original work [6] compares results of this method with numerical solutions for flows over wedges, cylinders, cylinders with suction, spheres, and the Schubauer ellipse. Only the results for skin friction over a wedge are presented here, compared with numerical calculations by Hartree [7]. These results are typical of the other geometries.

Table 1 presents comparisons in terms of dimensionless skin friction, $(C_f/2)\,Re_x^{1/2}$. The included wedge angle is given by $\pi\beta$, and results for a wide range of β's are shown. The potential flow for a given β is described by $u_e = x^m$, where $m = \beta/(2-\beta)$. Three terms of the solution produce errors of less than 2%. Two terms give results accurate to within approximately 5% except at very low or very high β. The simplicity of even the three-term solution allows results to be obtained immediately on a hand calculator.

Note that it is not necessary that the perturbation be performed about an ordinary differential equation (in the present case obtained from a similarity transformation). In the case of the boundary-layer equation, for example, it is possible to perform the perturbation directly by writing $\psi = \psi(y, \xi)$, $\xi = \xi(x)$ and then proceeding as before. This, in fact, produces reasonable results. The advantage of the similarity transformation is that more information is contained in the zero-order equation and convergence is thus more rapid.

Finally, although not specifically derived here, it is easy to obtain momentum thickness, displacement thickness, etc., from the analytical velocity solutions as presented. The thermal boundary-layer equation (constant property), which is linear, can now also be solved analytically by conventional means (or by this perturbation) by substitution of the appropriate velocity solutions into equation.

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